

# Weak Landau–Ginzburg models for smooth Fano threefolds

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To the memory of my teacher V. A. Iskovskikh.

**ABSTRACT.** We prove that Landau–Ginzburg models for all 17 smooth Fano threefolds with Picard rank 1 can be represented as Laurent polynomials in 3 variables exhibiting them case by case. We prove that general elements of all the pencils we found are birational to K3 surfaces. We state most of known methods of finding Landau–Ginzburg models in terms of Laurent polynomials. We state some problems concerning Laurent polynomial representations of Landau–Ginzburg models of Fano varieties.

## 1. INTRODUCTION

Mirror Symmetry conjectures relate symplectic properties of a variety  $X$  with algebro-geometric ones for its mirror symmetry pair — a variety  $Y$  (or for a pencil of Calabi–Yau varieties  $Y \rightarrow \mathbb{A}^1$ ), and vice-versa, relate algebro-geometric properties of  $X$  with symplectic ones of  $Y$ . One of the main problems in Mirror Symmetry is the problem of finding of such pairs for given variety. A pair for a smooth Fano variety  $X$  is conjecturally a so called *Landau–Ginzburg model*, that is, a (non-compact) manifold  $Y$  equipped with a complex-valued function called *potential*. The Mirror Symmetry conjecture of Hodge structures variations enables one to translate the mirror correspondence for Fano varieties to a quantitative level. In a lot of cases one may assume that  $Y$  is a complex torus. In this case the complex-valued function is represented by Laurent polynomial, which is called *a weak Landau–Ginzburg model* for  $X$ . It turns out that the problem of finding of weak Landau–Ginzburg model for  $X$  reduces to a finding Laurent polynomial such that the particular series combinatorially constructed by it (the so called *constant terms series*) equals the so called *constant term of regularized I-series* for  $X$  constructed by geometrical data (numbers of rational curves lying on  $X$ ). For more particular background see [Prz08].

In the paper we find weak Landau–Ginzburg models (some of them are known but had not been written down) for smooth Fano threefolds with Picard rank 1 (there are 17 families of such varieties, see [Isk77]; each of them is determined by its index and its anticanonical degree). It turns out that these models are Laurent polynomials in 3 variables and general elements of the pencils they define are birationally equivalent to K3 surfaces.

The paper is the first step of studying weak–Landau–Ginzburg models for Fano threefolds and their properties. We refer to [KP08] where some properties of some weak Landau–Ginzburg models and their relations with Homological Mirror Symmetry are studied.

We write down a table with weak Landau–Ginzburg models for Fano threefolds here for convenience. Later we prove that the polynomials from the table are actually weak Landau–Ginzburg models for corresponding Fano varieties (Theorem 14). We also observe

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The work was partially supported by FWF grant P20778, RFFI grants 08 – 01 – 00395-a and 06 – 01 – 72017–MNTI-a and grant NSh–1987.2008.1.

most of known methods of finding weak Landau–Ginzburg models (Section 3) and discuss some problems concerned with them (Section 5).

N.	Index	Degree	Description	Weak LG model
1	1	2	Sextic double solid $V'_2$ (a double cover of $\mathbb{P}^3$ ramified over smooth sextic).	$\frac{(x+y+z+1)^6}{xyz}$
2	1	4	The general element of the family is quartic.	$\frac{(x+y+z+1)^4}{xyz}$
3	1	6	A complete intersection of quadric and cubic.	$\frac{(x+1)^2(y+z+1)^3}{xyz}$
4	1	8	A complete intersection of three quadrics.	$\frac{(x+1)^2(y+1)^2(z+1)^2}{xyz}$
5	1	10	The general element is $V_{10}$ , a section of $G(2, 5)$ by 2 hyperplanes in Plücker embedding and quadric.	$\frac{(1+x+y+z+xy+xz+yz)^2}{xyz}$
6	1	12	The variety $V_{12}$ .	$\frac{(x+z+1)(x+y+z+1)(z+1)(y+z)}{xyz}$
7	1	14	The variety $V_{14}$ , a section of $G(2, 6)$ by 5 hyperplanes in Plücker embedding.	$\frac{(x+y+z+1)^2}{x} + \frac{(x+y+z+1)(y+z+1)(z+1)^2}{xyz}$
8	1	16	The variety $V_{16}$ .	$\frac{(x+y+z+1)(x+1)(y+1)(z+1)}{xyz}$
9	1	18	The variety $V_{18}$ .	$\frac{(x+y+z)(x+xz+xy+xyz+z+y+yz)}{xyz}$
10	1	22	The variety $V_{22}$ .	$\frac{xy}{z} + \frac{y}{z} + \frac{x}{z} + x + y + \frac{1}{z} + 4 + \frac{1}{x} + \frac{1}{y} + z + \frac{1}{xy} + \frac{z}{x} + \frac{z}{y} + \frac{z}{xy}$
11	2	$8 \cdot 1$	Double Veronese cone $V_1$ (a double cover of the cone over the Veronese surface branched in a smooth cubic).	$\frac{(x+y+1)^6}{xy^2z} + z$
12	2	$8 \cdot 2$	Quartic double solid $V_2$ (a double cover of $\mathbb{P}^3$ ramified over smooth quartic).	$\frac{(x+y+1)^4}{xyz} + z$
13	2	$8 \cdot 3$	A smooth cubic.	$\frac{(x+y+1)^3}{xyz} + z$
14	2	$8 \cdot 4$	A smooth intersection of two quadrics.	$\frac{(x+1)^2(y+1)^2}{xyz} + z$
15	2	$8 \cdot 5$	The variety $V_5$ , a section of $G(2, 5)$ by 3 hyperplanes in Plücker embedding.	$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz$
16	3	$27 \cdot 2$	A smooth quadric.	$\frac{(x+1)^2}{xyz} + y + z$
17	4	64	$\mathbb{P}^3$ .	$x + y + z + \frac{1}{xyz}$

TABLE 1. Weak Landau–Ginzburg models for Fano threefolds.

## 2. PRELIMINARIES

We consider smooth projective varieties over  $\mathbb{C}$ . For any such variety  $X$  we denote  $H_2(X, \mathbb{Z})/\text{tors}$  by  $H_2(X)$ . Calabi–Yau varieties in this paper mean varieties with trivial canonical class.

**2.1. Regularized quantum D-module.** Let  $X$  be a smooth Fano variety<sup>1</sup>. Then one may associate a set of Gromov–Witten invariants of genus 0 with it, that is, the numbers counting rational curves lying on  $X$ . Consider  $\gamma_1, \dots, \gamma_m \in H^*(X, \mathbb{Z})$ ,  $k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$ , and  $\beta \in H_2(X)$ . The genus 0 Gromov–Witten invariant with descendants that correspond to this data (see [Ma99], VI–2.1) we denote by

$$\langle \tau_{k_1} \gamma_1, \dots, \tau_{k_m} \gamma_m \rangle_{\beta}.$$

Given these invariants (more particular, prime three-pointed ones, that is, ones with  $m = 3$  and  $k_1 = k_2 = k_3 = 0$ ) one may define a (small) quantum cohomology ring (which is the deformation of ordinary cohomology ring).

**Definition 1** (see [Ma99], Definition 0.0.2). Consider the Novikov ring  $\Lambda$ , that is, the ring of polynomials over  $\mathbb{C}$  in formal variables  $t^{\beta}$ ,  $\beta \in H_2(X)$ , with natural relations  $t^{\beta_1} t^{\beta_2} = t^{\beta_1 + \beta_2}$ . Fix a basis  $\Delta \subset H^*(X, \mathbb{Z})$ . The *quantum cohomology ring* is a vector space  $QH^*(X) = H^*(X, \mathbb{Z}) \otimes \Lambda$  with *quantum multiplication*, i. e. a bilinear map

$$\star: QH^*(X) \times QH^*(X) \rightarrow QH^*(X)$$

given by

$$\gamma_1 \star \gamma_2 = \sum_{\substack{\gamma \in \Delta, \\ \beta \in H_2(X)}} t^{\beta} \langle \gamma_1, \gamma_2, \gamma^{\vee} \rangle_{\beta} \gamma$$

for any  $\gamma_1, \gamma_2 \in H^*(X)$ , where  $\gamma^{\vee}$  is the Poincaré dual class to  $\gamma$  (we identify elements  $\gamma \in H^*(X)$  and  $\gamma \otimes 1 \in QH^*(X)$ ).

Notice that  $QH^*(X)$  is graded with respect to the grading  $\deg t^{\beta} = -K_X \cdot \beta$  and the constant term of  $\gamma_1 \star \gamma_2$  (with respect to  $t$ ) is  $\gamma_1 \cdot \gamma_2$ , so  $QH^*(X)$  indeed is a deformation of  $H^*(X)$ .

Let  $H = -K_X$  and  $QH_H^*(X)$  be the minimal subring of  $QH^*(X)$  containing  $H$ . Let it be generated over  $\Lambda$  by the linear space  $H_H^*(X)$ , i. e.  $QH_H^* = H_H^*(X) \otimes \Lambda$ . The variety is called *quantum minimal* if  $\dim_{\Lambda} QH_H^* = \dim_{\mathbb{C}} H_H^*(X) = N + 1$ . The examples of quantum minimal varieties are complete intersections in (weighted) projective spaces or Fano threefolds with Picard rank 1. Now we describe the construction of *regularized quantum D-module*. More precisely, this  $\mathcal{D}$ -module contains an essential submodule corresponding to  $H_H^*(X)$ . As we need only this essential part, we give the definition of this submodule; one should replace  $H_H^*(X)$  by  $H^*(X)$  in the definition to get the definition of the whole module. For more particular definition of this  $\mathcal{D}$ -module for quantum minimal case see [Prz07], [Prz08], and [GS07].

Consider a torus  $\mathbb{T} = \text{Spec } B$ ,  $B = \mathbb{C}[t, t^{-1}]$ . Let  $HQ$  be a trivial vector bundle over  $\mathbb{T}$  with fiber  $H_H^*(X)$ . Let  $S = H^0(HQ)$  and let  $\star: S \times S \rightarrow S$  be the quantum multiplication (we may consider the quantum multiplication as an operation on  $S \cong QH_H^*(X) \otimes \mathbb{C}[t, t^{-1}]$ ).

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<sup>1</sup>This assumption may be generalized; we are interested in the case of smooth Fano varieties, so we give definitions in this particular case.

Let  $\mathcal{D} = B[t \frac{\partial}{\partial t}]$  and  $D = t \frac{\partial}{\partial t}$ . Consider a (flat) connection  $\nabla$  on  $HQ$  defined on the sections  $\gamma \in H_H^*(X)$  as

$$\left( \nabla(\gamma), t \frac{\partial}{\partial t} \right) = K_V \star \gamma$$

(the pairing is the natural pairing between differential forms and vector fields). This connection provides the structure of  $\mathcal{D}$ -module for  $S$  by  $D(\gamma) = (\nabla(\gamma), D)$ .

Let  $Q$  be this  $\mathcal{D}$ -module. It is not regular in general, so we need “to regularize” it to obtain the regular one. Let  $E = \mathcal{D}/\mathcal{D}(t \frac{\partial}{\partial t} - t)$  be the exponential  $\mathcal{D}$ -module. Define *the regularization* of  $Q$  as  $Q^{\text{reg}} = \mu_*(Q \boxtimes E)$ , where  $\mu: \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}$  is the multiplication and  $\boxtimes$  is the external tensor product (i. e.  $Q^{\text{reg}}$  is a convolution with the anticanonical exponential  $\mathcal{D}$ -module). It may be represented by a differential operator, which is divisible by  $D$  on the left:  $Q^{\text{reg}} \cong \mathcal{D}/\mathcal{D}(DL_X)$ . The differential operator  $L_X$  is called *regularized quantum differential operator*. If  $X$  is quantum minimal, then  $L_X$  is called *of type DN*, see [Go05], 2.10. This operator is explicitly written in [Go05], Example 2.11, for  $N = 3$  in terms of structural constants of quantum multiplication by the anticanonical class (that is, two-pointed Gromov–Witten invariants). Thus, there is an operator of type  $D3$  associated with every smooth Fano threefold with Picard group  $\mathbb{Z}$ . For all smooth Fano threefolds with Picard rank 1 the operators of type  $D3$  are known (see, for instance, [Go05], 5.8).

**Definition 2.** (A unique) analytic solution of  $L_X I = 0$  of type

$$I_{H^0}^X = 1 + a_1 t + a_2 t^2 + \dots \in \mathbb{C}[[t]], \quad a_i \in \mathbb{C},$$

is called *the fundamental term of the regularized  $I$ -series* of  $X$ .

Let  $H^0$  be the class in  $H^0(X, \mathbb{Z})$  dual to the fundamental class of quantum minimal variety  $X$ . Then this series is of the form

$$I_{H^0}^X = 1 + \sum_{\beta} \langle \tau_{-K_X \cdot \beta - 2} H^0 \rangle_{\beta} \cdot t^{-K_X \cdot \beta},$$

where the sum is taken over all  $\beta \in H_2(X)$  such that  $-K_X \cdot \beta \geq 2$  (see [Prz07], Corollary 2.2.6, and references therein).

**2.2. Weak Landau–Ginzburg model.** Consider a torus  $\mathbb{T}_{LG} = \mathbb{G}_{\mathbf{m}}^n = \prod_{i=1}^n \text{Spec } \mathbb{C}[x_i^{\pm 1}]$  and a function  $f$  on it. This function may be represented by Laurent polynomial:  $f = f(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$ . Let  $\phi_f(i)$  be the constant term (i. e. the coefficient at  $x_1^0 \dots x_n^0$ ) of  $f^i$ . Put

$$\Phi_f = \sum_{i=0}^{\infty} \phi_f(i) \cdot t^i \in \mathbb{C}[[t]].$$

**Definition 3.** The series  $\Phi_f = \sum_{i=0}^{\infty} \phi_f(i) \cdot t^i$  is called *the constant terms series* of  $f$ .

**Definition 4.** Let  $X$  be a smooth Fano variety and  $I_{H^0}^X \in \mathbb{C}[[t]]$  be its fundamental term of regularized  $I$ -series. The Laurent polynomial  $f \in \mathbb{C}[\mathbb{Z}^n]$  is called *a very weak Landau–Ginzburg model* for  $X$  if (up to the shift  $f \mapsto f + \alpha$ ,  $\alpha \in \mathbb{C}$ )

$$\Phi_f(t) = I_{H^0}^X(t).$$

The Laurent polynomial  $f \in \mathbb{C}[\mathbb{Z}^n]$  is called *a weak Landau–Ginzburg model* for  $X$  if it is a very weak Landau–Ginzburg model for  $X$  and for almost all  $\lambda \in \mathbb{C}$  the hypersurface  $\{1 - \lambda f = 0\}$  is birational to a Calabi–Yau variety.

The meaning of the definition is the following (see [BS85], 10, or [Be83], pp. 50–52). Consider a pencil  $\mathbb{T}_{LG} \rightarrow \mathbb{B} = \mathbb{P}[u : v] \setminus (0 : 1)$  with fibers  $Y_\alpha = \{1 - \alpha f = 0\}$ ,  $\alpha \in \mathbb{C} \setminus \{0\} \cup \{\infty\}$ .

The following proposition is a sort of a mathematical folklore (see [Prz08] for the proof).

**Proposition 5.** *Let the Newton polytope of  $f \in \mathbb{C}[\mathbb{Z}^n]$  contains 0 in the interior. Let  $t \in \mathbb{B}$  be the local coordinate around  $(0 : 1)$ . Then there is a fiberwise  $(n - 1)$ -form  $\omega_t \in \Omega_{\mathbb{T}_{LG}/\mathbb{B}}^{n-1}$  and (locally defined) fiberwise  $(n - 1)$ -cycle  $\Delta_t$  such that*

$$\Phi_f(t) = \int_{\Delta_t} \omega_t.$$

This means that  $\Phi_f(t)$  is a solution of the Picard–Fuchs equation for the pencil  $\{Y_t\}$ .

*Remark 6.* Let  $PF_f = PF_f(t, \frac{\partial}{\partial t})$  be a Picard–Fuchs operator of  $\{Y_t\}$ . Let  $m$  be the order of  $PF_f$  and  $r$  be the degree with respect to  $t$ . Let  $Y$  be a semistable compactification of  $\{Y_t\}$  (so we have the map  $\tilde{f}: Y \rightarrow \mathbb{P}^1$ ; denote it for simplicity by  $f$ ). Let  $m_f$  be the dimension of transcendental part of  $R^{n-1}f_! \mathbb{Z}_Y$  (for the algorithm for computing it see [DH86]). Let  $r_f$  be the number of singularities of  $f$  (counted with multiplicities). Then  $m \leq m_f$  and  $r \leq r_f$ . So we may write a differential operator of bounded order by  $t$  and  $D$  with indeterminant coefficients and, as  $\Phi_f$  vanishes it, get a system of infinite number of linear equations. To check that  $L_X = PF_f$  we need to solve this system (it has a unique, up to scaling, solution, so we need to solve the finite system of linear equations).

However in practice it is enough to compare the first few coefficients of the expansion of  $\Phi_f$  and  $I_{H^0}^X$ . As  $PF_f \Phi = 0$  has a unique solution then the first few coefficients  $\Phi_f$  determine the other ones. The minimal number of this terms (denote it by  $s$ ) a priori depends on  $f$ ; in practice  $s$  is as minimal as possible (that is,  $mn - 1$ ). As the analytical solution of  $L_X I = 0$  is determined by the first  $\dim X = m$  coefficients, then to prove that  $I_{H^0}^X = \Phi_f$  we need to check only first  $s$  coefficients.

**Question 7.** *Let us be given a polytope  $\Delta$ . Find (effectively) a number  $s = s(\Delta)$  such that for any polynomial  $f$  whose Newton polytope is  $\Delta$  the first  $s$  coefficients of  $\Phi_f$  determine the rest ones (so to prove that  $I_{H^0}^X = \Phi_f$  for any  $I_{H^0}^X$  it is enough to check coincidence of the first  $s$  coefficients of both series). In the other words, is it true that the linear system for indeterminant coefficients of a Picard–Fuchs equation for Laurent polynomial with given Newton polytope for first  $s$  coefficients is nondegenerate?*

### 3. METHODS OF FINDING OF WEAK LANDAU–GINZBURG MODELS

We observe here some methods of finding very weak Landau–Ginzburg models for some Fano varieties, that is, complete intersections in projective spaces and Grassmannians and varieties admitting small toric degenerations. They are weak ones in practice, that is, their general fibers are birational to Calabi–Yau varieties and usually it is not complicated to prove it; however, we do not know the general method of proving this in some cases.

In the next section we find weak Landau–Ginzburg models for Fano threefolds with Picard number 1.

**3.1. Small toric degenerations.** We start from description of mirrors for Fano varieties admitting the so called small toric degenerations. These descriptions were suggested in [BCFKS97] and [Ba97].

Let the smooth Fano variety  $X$  admit a degeneration to terminal Gorenstein toric variety  $Y$ . Let  $\{v_1, \dots, v_n\} \subset \mathbb{Z}^k$ ,  $v_i = (v_i^1, \dots, v_i^k)$  be the set of integral generators of rays of a fan of  $Y$ . Denote  $x^{v_i} = x_1^{v_i^1} \cdot \dots \cdot x_k^{v_i^k} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$ . Then the weak Landau–Ginzburg model for  $X$  (up to the shift  $f \rightarrow f + \alpha$ ,  $\alpha \in \mathbb{C}$ ) is

$$\sum_{i=1}^n x^{v_i}.$$

**Problem 8.** *Prove this. One may use Quantum Lefschetz–type arguments (applied to hyperplane section) and Givental’s formula for  $I$ -series of smooth toric varieties (cf. [Ga07], Proposition 1.7.15) for proving that these polynomials are very weak Landau–Ginzburg models. For proving that general elements of corresponding pencils are birational to Calabi–Yau varieties one should check that singularities of these general elements either admit a crepant resolution or “come from the ambient toric variety” (this is enough for the proof, because these elements are anticanonical sections of ambient toric variety). For references see [Ba93], in particular, Theorem 4.1.9.*

**3.2. Complete intersections.** The suggestions for Landau–Ginzburg models for complete intersections in projective spaces were given by Hori and Vafa in [HV00]. In terms of Laurent polynomials their suggestions may be stated in the following way. The weak Landau–Ginzburg model for smooth complete intersection  $X$  of hypersurfaces of degrees  $k_1, \dots, k_r$  in  $\mathbb{P}^N$  is (up to the shift  $f \rightarrow f + \alpha$ ,  $\alpha \in \mathbb{C}$ )

$$f_X = \frac{\prod_{i=1}^r (x_{i,1} + \dots + x_{i,k_i-1} + 1)^{k_i}}{\prod x_{i,j} \cdot \prod y_i} + y_1 + \dots + y_{k_0} \in \mathbb{C}[\{x_j, x_j^{-1}, y_s, y_s^{-1}\}],$$

where  $k_0 = N - \sum k_i$ .

**Proposition 9.** *The polynomial  $f_X$  is a weak Landau–Ginzburg model for  $X$ .*

**Proof.** According to well-known Givental’s formula for constant term of  $I$ -series of  $X$  (up to the shift),

$$I_{H^0}^X = \sum_{i=0}^{\infty} \frac{\prod_{j=0}^r (k_j i)!}{(i!)^{N+1}} t^{k_0 i}.$$

One may easily check that the constant term of  $f_X^{ik_0}$  is exactly  $\frac{\prod_{j=0}^r (k_j i)!}{(i!)^{N+1}}$  and 0 for the other degrees.

The compactification of the pencil corresponding to  $f_X$  given by the natural embedding  $(\mathbb{C}^*)^{N-r} \hookrightarrow \mathbb{P}(y_0 : \dots : y_{k_0}) \times \mathbb{P}(x_{1,1} : \dots : x_{1,k_1}) \times \dots \times \mathbb{P}(x_{r,1} : \dots : x_{r,k_r})$  shows that the general element of the pencil is birational to (singular) hypersurface of multidegree  $(k_0 + 1, k_1, \dots, k_r)$  in  $\mathbb{P}^{k_0} \times \mathbb{P}^{k_1-1} \times \dots \times \mathbb{P}^{k_r-1}$  and hence has trivial canonical class. It is easy to check that the singularities of the general fiber are “pure canonical”. More precisely, elements of these pencils admit crepant resolutions. Indeed, they are products of du Val singularities and linear spaces (away from intersections of components of singularities). After blowing up one of intersected components another one is du Val singularity along a linear space in the neighborhood of an exceptional divisor (singularities “intersect transversally”). Hence a general fiber is birational to Calabi–Yau variety. Thus the Laurent polynomials we consider are actually weak Landau–Ginzburg models.  $\square$

**3.3. Grassmannians.** Weak Landau–Ginzburg models for Grassmannians were suggested in [EHX97], B 25. Later in [BCFKS97], using the construction of small toric degenerations of Grassmannians (see references therein), these Landau–Ginzburg models were obtained via small toric degenerations technics.

The suggested model for  $X = G(k, N)$  is

$$X_{1,1} + \sum_{\substack{1 \leq a \leq N-k, \\ 1 \leq b \leq k}} X_{ab}^{-1} (X_{a+1,b} + X_{a,b+1}) + X_{N-k,k}^{-1} \in \mathbb{C}[\{X_{ab}, X_{ab}^{-1}\}]$$

(the variables are  $X_{a,b}$ ,  $1 \leq a \leq N-k$ ,  $1 \leq b \leq k$ ; for  $a > N-k$  or  $b > k$  we put  $X_{ab} = 0$ ).

**Problem 10.** *Prove that the polynomial written down above is actually weak (or at least very weak) Landau–Ginzburg model for  $G(k, N)$ . The problem is combinatorial: to solve it one should find the coefficients of constant term series for this polynomial and compare it with constant term of regularized  $I$ -series for  $G(k, N)$  found in [BCK03] (see also [BCFKS98], Conjecture 5.2.3). Another, more conceptual way, suggested by S. Galkin, is the following (see [Ga07]). Consider the general hyperplane section  $Y$  in Grassmannian. Its constant term of regularized  $I$ -series should be equal to the constant term of regularized  $I$ -series of its deformation  $Y_0$ , that is, a hypersurface in toric terminal variety  $X$ . Consider a small toric resolution  $\tilde{X}$  of this variety. Then constant term of regularized  $I$ -series of this variety equals to the constant terms series of corresponding Laurent polynomial (see [Ga07]). As  $Y_0$  do not intersect the singular locus of  $X$  (we use the fact that  $X$  is Gorenstein here), the Quantum Lefschetz Theorem applied to  $Y_0$  lying in  $\tilde{X}$  is similar to Quantum Lefschetz Theorem applied to  $Y$  lying in Grassmannian (see [Prz05]). As a procedure of Quantum Lefschetz Theorem is invertible in this case we may conclude that the constant term of  $I$ -series of Grassmannian equals the constant term series of the corresponding polynomial. For methods of proving of Calabi–Yau condition see Problem 8.*

**3.4. Complete intersections in Grassmannians.** In this section we describe the suggestions for weak Landau–Ginzburg models of complete intersections in Grassmannians. The idea for writing them down is the particular case of the method of writing down of Landau–Ginzburg models for complete intersections in varieties admitting small toric degenerations. More particularly, let  $Y = X \cap Y_1 \cap \dots \cap Y_s$  be a complete intersection in a variety admitting a degeneration to terminal Gorenstein toric variety  $X$ . Let  $\{v_1, \dots, v_n\} \subset \mathbb{Z}^k$  be the set of integral generators of rays of a fan of  $X$  as before. Let  $\{p_1^l, \dots, p_{r_l}^l\}$ ,  $l = 1, \dots, s$ , be subsets of  $\{v_1, \dots, v_n\}$  such that  $Y_l = \sum_j p_j^l$  as cohomological classes. Then the Landau–Ginzburg model dual to  $Y$  is (conjecturally)

$$\{x^{p_1^l} + \dots + x^{p_{r_l}^l} = 1\} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_k^{\pm 1}], \quad l = 1, \dots, s,$$

with potential  $\sum_{i=1}^n x^{v_i}$ .

We describe this procedure for complete intersections in Grassmannian  $G(m, r)$  following [BCFKS97] (and changing the notations for simplicity)<sup>2</sup>. As the Picard rank of Grassmannian is 1, the cohomological class of hypersurface therein is given by its degree. So it is enough to describe the subsets of generators of rays of the fan of toric degeneration of Grassmannian described above such that the sum of vectors in each subset is a generator of the Picard group of this variety. The corresponding Laurent polynomials, that is,

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<sup>2</sup>It was suggested for Calabi–Yau complete intersections but it works for Fano varieties in absolutely the same way (as usual, modulo shift  $f \rightarrow f + \alpha$ ,  $\alpha \in \mathbb{C}$ ).

polynomials  $x^{p_1^i} + \dots + x^{p_{r_i}^i}$ ,  $i = 1, \dots, n$ , in variables  $X_{ab}$ ,  $1 \leq a \leq r - m$ ,  $1 \leq b \leq r$ , are the following.

$$\begin{aligned} & X_{11}, \\ & \frac{X_{1i}}{X_{1,i-1}} + \frac{X_{2i}}{X_{2,i-1}} + \dots + \frac{X_{r-m,i}}{X_{r-m,i-1}}, \quad i = 2, \dots, m, \\ & \frac{X_{j+1,1}}{X_{j1}} + \frac{X_{j+1,2}}{X_{j2}} + \dots + \frac{X_{j+1,m}}{X_{jm}}, \quad j = 1, \dots, r - m - 1, \\ & \frac{1}{X_{r-m,k}}. \end{aligned}$$

It turns out that the procedure described above may be done on the level of Laurent polynomials for the case of complete intersections in Grassmannians. More particularly, the following proposition holds.

**Proposition 11.** *There is the (birational) change of variables for the Landau–Ginzburg model for Fano complete intersection in  $G(m, r)$  obtained in the way described above, such that after this change Landau–Ginzburg model is a function on complex torus (that is, a Laurent polynomial).*

*Proof.* Express one variable from each equation, put them in the Laurent polynomial of Grassmannian and make linear change of variables to make denominators monomial.  $\square$

It does not follow from this procedure that the Laurent polynomial obtained in this way is a very weak Landau–Ginzburg model for complete intersection in  $G(m, r)$ . It is even a weak one in fact; it is easy to check this in each particular case.

**Problem 12.** *Prove this in the general case (cf. Problems 8 and 10).*

**Example 13.** Consider the Fano threefold  $V_{14}$ , that is, the section of  $G(2, 6)$  by five hyperplanes (see [Isk77]). The Landau–Ginzburg model is the variety

$$\left\{ X_{11} = 1, \frac{X_{21}}{X_{11}} + \frac{X_{22}}{X_{12}} = 1, \frac{X_{31}}{X_{21}} + \frac{X_{32}}{X_{22}} = 1, \frac{X_{41}}{X_{31}} + \frac{X_{42}}{X_{32}} = 1, \frac{1}{X_{42}} = 1 \right\} \subset \mathbb{C}[\{X_{ij}, X_{ij}^{-1}\}],$$

$1 \leq i \leq 4$ ,  $1 \leq j \leq 2$ , with potential

$$X_{11} + \frac{X_{21} + X_{12}}{X_{11}} + \frac{X_{22}}{X_{12}} + \frac{X_{31} + X_{22}}{X_{21}} + \frac{X_{32}}{X_{22}} + \frac{X_{41} + X_{32}}{X_{31}} + \frac{X_{42}}{X_{32}} + \frac{X_{42}}{X_{41}} + \frac{1}{X_{42}}.$$

Denote  $X_{12} = a$ ,  $X_{22} = b$ ,  $X_{32} = c$ . Then

$$\begin{aligned} X_{21} &= \frac{a - b}{c}, \\ X_{31} &= \frac{(a - b)(b - c)}{ab}, \\ X_{41} &= \frac{(c - 1)(a - b)(b - c)}{abc}. \end{aligned}$$

So the potential is

$$5 + a + \frac{ab}{a - b} + \frac{abc}{(a - b)(b - c)} + \frac{abc}{(a - b)(b - c)(c - 1)} = 5 + \frac{a^2}{a - b} + \frac{abc^2}{(a - b)(b - c)(c - 1)}.$$



Denote  $x = a - b$ ,  $y = b - c$ ,  $z = c - 1$ . Then  $a = x + y + z + 1$ ,  $b = y + z + 1$ ,  $c = z + 1$  and we get the potential

$$f = 5 + \frac{(x + y + z + 1)^2}{x} + \frac{(x + y + z + 1)(y + z + 1)(z + 1)^2}{xyz} \in \mathbb{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}].$$

The constant term of regularized  $I$ -series for  $V_{14}$  (shifted by 4) is

$$I_{H^0}^{V_{14}} = 1 + 4t + 48t^2 + 760t^3 + 13840t^4 + 273504t^5 + 5703096t^6 + \dots$$

(see [Prz04]). It is easy to see that the constant terms series for  $f_{14} = f - 5$  equals  $I_{H^0}^{V_{14}}$  up to more then 16 coefficients. This means that they are equal (see Remark 6).

The standard compactification of the general element of the pencil  $f = \lambda$ ,  $\lambda \in \mathbb{C}$ , given by the natural map  $\text{Spec } \mathbb{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}] \rightarrow \mathbb{P}(x : y : z : w)$  is quartic surface with du Val singularities, so it is birational to a K3 surface.

So, the Laurent polynomial we get is a weak Landau–Ginzburg model for  $V_{14}$ .

#### 4. FANO THREEFOLDS OF RANK 1

In this section we study Laurent polynomials from Table 1 and prove that they are weak Landau–Ginzburg models for corresponding Fano varieties. However, first we describe how these weak Landau–Ginzburg models were obtained.

Varieties **1**, **11**, **12** are hypersurfaces in weighted projective spaces (of degree 6 in  $\mathbb{P}(1 : 1 : 1 : 1 : 3)$ , of degree 6 in  $\mathbb{P}(1 : 1 : 1 : 1 : 2 : 3)$ , and of degree 4 in  $\mathbb{P}(1 : 1 : 1 : 1 : 1 : 2)$  respectively). Their weak Landau–Ginzburg models may be found by Hori–Vafa procedure similar to procedure for complete intersections in projective spaces described in 3.2 (see the proof of Theorem 14).

Varieties **2**, **3**, **4**, **13**, **14**, **16**, **17** are complete intersections, so they may be found using Proposition 9.

Varieties **5**, **7**, **15** are complete intersections in Grassmannians, so the corresponding polynomials may be obtained using Proposition 11. The polynomial for  $V_{14}$  is studied in Example 13. There is another way to obtain the same polynomials for  $V_5$  and  $V_{10}$ . That is,  $V_5$  has a small toric degeneration (this is proved by S. Galkin in his Thesis [Ga07]), so its weak Landau–Ginzburg model is given by the corresponding polytope (see 3.1). According to Golyshev (see [Go05]), the Landau–Ginzburg model for  $V_{10}$  is a quotient of the model for  $V_5$ . Taking invariants of quotient and changing coordinates one can get its weak Landau–Ginzburg model; the form we wrote down is convenient for calculations.

The polynomial for **6**-th variety  $V_{12}$  was found in [BP84]. We change coordinates a bit to get the convenient form as written.

Finally, polynomials for varieties **8**, **9**, **10** were found in [Prz08]. There is a misprint in the polynomial for  $V_{16}$  in the journal version of [Prz08]; it is corrected in the preprint on arXiv. It is remarkable that some of these polynomials were found under assumption that there are canonical Gorenstein toric degenerations of corresponding varieties. Later S. Galkin in his Thesis ([Ga07]) proved that there is a terminal Gorenstein toric degeneration of  $V_{22}$ , so the corresponding polynomial may be obtained using a method from 3.1.

**Theorem 14.** *The polynomials from Table 1 are weak Landau–Ginzburg models for Fano threefolds with Picard rank 1.*

**Proof.** Direct computations show that these polynomials are very weak Landau–Ginzburg models (see Remark 6). Moreover, general elements of pencils corresponding to

all polynomials except for polynomials for  $V_1$  and  $V'_2$  after compactification are quartics in  $\mathbb{P}^3$  or (for complete intersections) anticanonical sections in products of projective spaces with du Val singularities (cf. Proposition 9), so they are birationally isomorphic to K3 surfaces.

Let us prove that a general element of the pencil for  $V'_2$  is birational to a K3 surface. The standard compactification gives the pencil of non-normal sextics in  $\mathbb{P}^3$ . The singularities of these sextics are du Val points and the line. One may blow up this line (2 times) to get a surface with du Val singularities and check that the canonical class is trivial. However we do it in a more conceptual way (suggested by V. Golyshev). Remember that a Hori–Vafa mirror for a hypersurface of degree  $d$  in  $\mathbb{P}(w_0 : \dots : w_n)$  is

$$\begin{cases} y_0^{w_0} \cdot \dots \cdot y_n^{w_n} = 1 \\ y_0 + \dots + y_k = 1, \end{cases}$$

where  $w_0 + \dots + w_k = d$ , with potential

$$f = y_0 + \dots + y_n.$$

For  $V'_2$ , i. e. for the hypersurface of degree 6 in  $\mathbb{P}(1 : 1 : 1 : 1 : 3)$  we have (up to shift  $f \rightarrow f - 1$ ) the variety

$$\begin{cases} y_0 y_1 y_2 y_3 y_4^3 = 1 \\ y_1 + y_2 + y_3 + y_4 = 1 \end{cases}$$

with potential

$$f = y_0.$$

Taking change of variables

$$y_1 = \frac{x}{x + y + z + t}, \quad y_2 = \frac{y}{x + y + z + t}, \quad y_3 = \frac{z}{x + y + z + t}, \quad y_4 = \frac{t}{x + y + z + t}$$

(where  $x, y, z, t$  are projective coordinates) we get the Landau–Ginzburg model

$$y_0 x y z t^3 = (x + y + z + t)^6, \quad f = y_0.$$

So, in local map, say,  $t \neq 0$  we finally get the weak Landau–Ginzburg model

$$f'_2 = \frac{(x + y + z + 1)^6}{xyz}.$$

The general element of the pencil corresponding to  $f'_2$  is the general element of the initial Hori–Vafa model. Inverse the potential:  $u = 1/f$ . Then we get the pencil

$$y_1 y_2 y_3 y_4^3 = u, \quad y_1 + y_2 + y_3 + y_4 = 1.$$

This model is exactly the Landau–Ginzburg model for  $\mathbb{P}(1 : 1 : 1 : 3)$  (see [CG06], (2)). So, by Theorem 1.15 in [CG06], the general element of the pencil we are interested in is birational to a K3 surface.

The similar may be done for  $V_1$ . Interestingly, there are two ways to decompose the degree to a sum of weights for this case, that is, for hypersurface of degree 6 in  $\mathbb{P}(1 : 1 : 1 : 2 : 3)$ :  $6 = 1 + 1 + 1 + 3$  and  $6 = 1 + 2 + 3$ . The change similar to one described above may be done only for the second of them (otherwise we need to take a square root of a Laurent polynomial). Finally prove that the general fiber of the pencil

$$\frac{(x + y + 1)^6}{xy^2 z} + z = \lambda, \quad \lambda \in \mathbb{C},$$

associated with  $V_1$  is birational to a K3 surface. Compactify it to the surface in  $\mathbb{A}^3$ :

$$(x + y + 1)^6 = (\lambda - z)xy^2z.$$

Then change the variables  $a = x + y + 1$ . We get

$$a^6 = (\lambda - z)(a - y - 1)y^2z.$$

Return to the torus: consider this surface as lying in  $(\mathbb{C}^*)^3$  with coordinates  $a, x, y$  (that is, let us divide by  $a, x, y$ ). Change the variables:  $b = y/a, c = yz/a$ . We get the surface

$$a^4 = (\lambda b - c)(a - ab - 1)c.$$

Compactify it as lying in  $\mathbb{P}^3$ . We get a quartic with du Val singularities, whose resolution is a K3 surface.  $\square$

## 5. QUESTIONS

We state few problems in this section.

All varieties we discuss in this paper have weak Landau–Ginzburg models of the same dimension. So it is natural to state the following conjecture.

**Conjecture 15.** *Any smooth Fano variety of dimension  $N$  with Picard rank 1 has a weak Landau–Ginzburg model  $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$ .*

*Remark 16.* Assuming that a Landau–Ginzburg model is a compactification of a weak one we get the following immediate corollary of this conjecture: for any smooth Fano variety with Picard rank 1 there is a rational Landau–Ginzburg model.

**Question 17.** *This conjecture seems to hold also for quantum minimal varieties. There are two non-quantum minimal del Pezzo surfaces: ones of degrees 7 and 8. However, Landau–Ginzburg models for them (found in [AKO05]) are rational. Is it true that Conjecture 15 may be generalized at least in the following weak form: for any smooth Fano variety there is a rational Landau–Ginzburg model (corresponding to the anticanonical divisor or, more general, to any one)?*

Of course, a Fano variety may have several weak Landau–Ginzburg models. For instance, there are at least two different polynomials for Fano threefold  $V_1$ . They are given by different ways of application of Hori–Vafa procedure for complete intersections in weighted projective spaces (see the proof of Theorem 14). That is,

$$\frac{(x^2 + y^2 + z^2 + 1)^3}{xyz} \in \mathbb{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$$

and

$$\frac{(x + y + 1)^6}{xy^2z} + z \in \mathbb{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}].$$

Another example are complete intersections, which have weak Landau–Ginzburg models described in 3.2 and have weak Landau–Ginzburg models — products of polynomials for projective spaces just by combinatorial reasons. Thus a 4-dimensional cubic (see [KP08]) has at least two weak Landau–Ginzburg models:

$$\frac{(x + y + 1)^3}{xyzw} + z + w$$

and

$$\left(x_1 + x_2 + \frac{1}{x_1x_2}\right) \left(y_1 + y_2 + \frac{1}{y_1y_2}\right).$$

The Hori–Vafa-type polynomials look “more natural”. So the challenging problem is to find “correct” weak Landau–Ginzburg model for given Fano variety  $X$ .

This problem seems to be concerned with toric degenerations. For instance, if a weak Landau–Ginzburg model for Fano threefold  $X$  has a Newton polytope corresponding to a terminal Gorenstein toric variety with the same numerical data (the degree, the Picard rank, etc.), then  $X$  has a degeneration to this toric variety (see [Ga07]). Newton polytopes of weak Landau–Ginzburg models from Table 1 correspond to toric varieties (mostly Gorenstein) also with the same numerical data as the numerical data for the initial varieties. So it is natural to assume the existence of a generalization of the method from 3.1 (cf. [Prz08], 4.3.1 and 4.3.2 and discussion therein).

**Definition 18.** Let  $X$  be a smooth Fano variety of dimension  $N$ . Then a Laurent polynomial  $f$  in  $N$  variables is called a *semiweak Landau–Ginzburg model* for  $X$  if  $f$  is a weak Landau–Ginzburg model for  $X$  and a volume of the polytope dual to the Newton polytope of  $f$  equals  $(-K_X)^N/N!$ .

*Remark 19* (S. Galkin). Consider a smooth Fano variety  $X$ . Then it has a projective degeneration to toric variety  $T$  only if the anticanonical Hilbert polynomial for  $X$  equals Ehrhart polynomial for the polytope corresponding to  $T$  (that is, dual to the convex hull of the generators of rays of a fan for  $T$ ). So the degree condition in Definition 18 may be generalized to the following condition: the Hilbert polynomial of  $X$  equals the Ehrhart polynomial of a polytope dual to the Newton polytope of  $f$ .

**Question 20.** Let  $f$  be a semiweak Landau–Ginzburg model for Fano variety  $X$ . Let  $T$  be a toric variety such that the linear span of integral generators of rays of a fan of  $T$  is the Newton polytope of  $f$ . Does  $X$  degenerate to  $T$ ? Find the conditions on toric varieties (like a type of singularities) such that semiweak Landau–Ginzburg models with support on polytopes corresponding to these toric varieties are in 1–1 correspondence with degenerations of  $X$  to these toric varieties.

*Remark 21.* In particular, there is a problem set by V. Golyshev. Characterize those Fano varieties that admit a weak Landau–Ginzburg models supported on a polytope with a single strictly internal point.

The last question is motivated by the following observation. A Newton polytope of weak Landau–Ginzburg model seems to “know” the Laurent polynomial. That is, the coefficients at vertices of Newton polytope for Fano threefolds equal 1, the coefficient at  $m$ -th integral point form the edge of length  $k + 1$  is  $\binom{k}{m}$ , etc.

**Question 22.** Suppose we know a weak Landau–Ginzburg model  $f$  for Fano variety. Let  $\Delta$  be its Newton polytope. Is there an effective algorithm for determining  $f$  in terms of  $\Delta$ ?

The author is grateful V. Golyshev, A. Iliev, L. Katzarkov, D. Orlov, K. Shramov, and A. Wilson for helpful comments and to S. Galkin for important remarks.

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